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ELASTIC VIBRATIONS OF PARABOLOIDAL

SHELLS OF REVOLUTION

by

W. H. Hoppmann II, M. I. Cohen, and

V. X. Kunukkasseril

Department of Mechanics
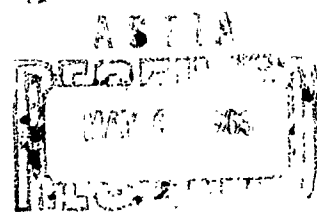
Rensselaer Polytechnic Institute

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Abstract

In this report there is presented a discussion of the
theory of vibration of a paraboloidal shell of revolution.
The theory is that which is usually associated with the
work of A. E. H. Love. Observations are made concerning
the equations of motion which include both flexural and
membrane stresses, those which relate to flexural and mem-
brane stresses for shallow shells, and finally those which
apply to membrane stresses only.

Results are also presented for an extensive experimen-
tal study of the vibrations of two models of thin parabo-
loidal shells.

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α, β	=	surface coordinates
A, B	=	metric coefficients associated with the parametric curves, α = constant and β = constant
Θ	=	phase angle
ϵ_1, ϵ_2	=	lineal strain components
R_1, R_2	=	radii of principal curvature
Φ	$\equiv$	α
r, z	=	coordinates of points on paraboloidal surface
a	=	scale factor of paraboloid
T_1, T_2	=	components of stress resultants corresponding to extensions in coordinate directions
G_1, G_2	=	components of stress couples corresponding to coordinate directions
S_1, S_2	=	components of shearing stress resultants in tangent plane at a point and corresponding to coordinate directions
N_1, N_2	=	components of shear stress resultants in direction normal to the surface
u	=	displacement in latitudinal direction
v	=	displacement in meridional direction
w	=	displacement in direction normal to surface
p_n	=	natural frequency of nth mode
ρ	=	mass density of shell
h	=	thickness of shell
E	=	Young's modulus
ν	=	Poisson's ratio
K_1, K_2	=	curvatures corresponding to principal directions
s_1	=	distance on surface in latitudinal direction
s_2	=	distance on surface in meridional direction

INTRODUCTION

In the technical literature there is an extreme paucity of complete solutions of even the approximate equations of motion of shells developed by A. E. H. Love. A few examples may be cited such as solutions by E. Reissner [1][4], W. H. Hoppmann [2] and Naghdi and Kalnins [3]. As is well known the reason for the difficulty lies in the complexity of the differential equations defining the motion. Reasonably tractable series solutions simply cannot be found. Consequently it appears that reliable numerical results must depend on the programming of suitable computers for specific cases of interest. However, it will probably always be true that analyses of the equations of motion subject to various boundary conditions along with careful experimental determinations of modal shapes and frequencies for various shapes of shells will be necessary for building up reliable technological information concerning such problems.

In the present report a study is made of the vibrations of paraboloidal shells of revolution. A development of equations of motion is given along with experimental results for two models of paraboloidal shells.

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[4] Numbers in brackets designate References at end of Report.

GEOMETRICAL CONSIDERATIONS

The meridional cross-section of a paraboloid of revolution shown in Fig. 1 is a parabola defined by

$$r^2 = az \quad . \tag{1}$$

The metric may be written

$$
\begin{aligned}
ds^2 &= ds_1^2 + ds_2^2 \\
&= A^2 d\alpha^2 + B^2 d\beta^2 \tag{2} \\
&= (dr^2 + dz^2) + r^2 d\beta^2
\end{aligned}
$$

whence $\quad A = \sqrt{1 + \dfrac{a}{4z}}$

$$\tag{3}$$

and $\quad B = r = \sqrt{az} \quad .$

The principal radii of curvature may be written

$$R_1 = \frac{(4z + a)^{3/2}}{2\sqrt{a}}$$

$$\tag{4}$$

and $\quad R_2 = \sqrt{az + \dfrac{a^2}{4}} \quad .$

EQUATIONS OF FLEXURAL VIBRATIONS

The analysis will now be limited to the case of axisymmetric vibrations and the equations of motion developed by Love may be written [4]

$$\frac{\partial(T_1 B)}{\partial z} - T_2 \frac{\partial B}{\partial z} - N_1 \frac{AB}{R_1} - AB\rho h \ddot{v} = 0 \quad ,$$

$$N_2 \frac{AB}{R^2} + S_2 \frac{\partial B}{\partial z} = 0 \quad ,$$

$$\frac{\partial(N_1 B)}{\partial z} + T_1 \frac{AB}{R_1} + T_2 \frac{AB}{R_2} - AB\rho h \ddot{w} = 0 \quad ,$$

$$N_2 = 0 \quad ,$$

$$\frac{\partial(G_1 B)}{\partial z} - G_2 \frac{\partial B}{\partial z} - N_1 AB = 0 \quad .$$

and $\quad S_2 = 0 \quad .$

These equations reduce to

$$(T_1 - T_2) \frac{\partial B}{\partial z} + B \frac{\partial T_1}{\partial z} + (G_2 - G_1) \frac{1}{R_1} \frac{\partial B}{\partial z} - \frac{B}{R_1} \frac{\partial G_1}{\partial z} - AB\rho h \ddot{v} = 0$$

and

$$\frac{T_1}{R_1} + \frac{T_2}{R_2} + \frac{1}{AB} \frac{\partial}{\partial z} \left\{ \frac{1}{A} \frac{\partial(G_1 B)}{\partial z} - \frac{1}{A} G_2 \frac{\partial B}{\partial z} \right\} - \rho h \ddot{w} = 0 \quad .$$

(6)

Since $r = R_2 \sin \alpha$ Equations (6) may be written

$$(T_1 - T_2) \cot \alpha + T_1' \cos^2 \alpha + \frac{2}{a} \frac{\cos^4 \alpha}{\sin \alpha} (G_2 - G_1)$$

$$- \frac{2}{a} \cos^5 \alpha \, G_1' - \frac{a}{2} \sec \alpha \, \rho h \ddot{v} = 0$$

and   (7)

$$T_1 \cos^3 \alpha + T_2 \cos \alpha + \frac{2}{a} \frac{\cos^4 \alpha}{\sin \alpha} \left\{ \cos^3 \alpha (G_1 \tan \alpha)' \right.$$

$$\left. - \cos \alpha \, G_2 \right\}' - \frac{a}{2} \rho h \ddot{w} = 0$$

where $(\ )'$ refers to derivatives with respect to α.

The relations between strains and displacements may be written in accordance with Love [4] as

$$\epsilon_1 = \frac{1}{A} \frac{dv}{dz} - \frac{w}{R_1} \qquad , \qquad \epsilon_2 = \frac{v}{AB} \frac{\partial B}{\partial z} - \frac{w}{R_2}$$

$$K_1 = \frac{1}{A} \frac{\partial}{\partial z}(\frac{1}{A} \frac{\partial w}{\partial z} + \frac{v}{R_1}) \qquad , \text{ and } \quad K_2 = \frac{1}{AB} \frac{\partial B}{\partial z}(\frac{1}{A} \frac{\partial w}{\partial z} + \frac{v}{R_1}) \quad .$$

(8)

The stress-strain relations for isotropic material are taken as

$$T_1 = \frac{Eh}{1-\nu^2} (\epsilon_1 + \nu \epsilon_2) \qquad , \qquad T_2 = \frac{Eh}{1-\nu^2} (\epsilon_2 + \nu \epsilon_1)$$

(9)

$$G_1 = -D(K_1 + \nu K_2) \qquad , \text{ and } \quad G_2 = -D(K_2 + \nu K_1) \quad .$$

-6-

Considering Equations (7), (8), and (9) simultaneously the equations of motion may be written

$$a_1 w''' + a_2 w'' + a_3 w' + a_4 w + a_5 v'' + a_6 v' + a_7 v = a_8 \ddot{v}$$

and   (10)

$$b_1 w'''' + b_2 w''' + b_3 w'' + b_4 w' + b_5 w + b_6 v''' + b_7 v''$$

$$+ b_8 v' + b_9 v = b_{10} \ddot{w}$$

where the a_i's and b_i's are determined functions of α.

The complexity of these equations is obvious and it is clear that at present an extensive programming operation for high speed computers would be necessary to tabulate the solutions.

Considerable simplifications in these equations result if only shallow paraboloidal shells are considered. Such simplifications will now be considered.

FLEXURAL VIBRATIONS OF SHALLOW PARABOLOIDAL SHELLS OF REVOLUTION

If the shell shown in Fig. 1 is truncated so that its base is given by $\alpha = \Phi_0$ and

$$\sin \Phi_0 \doteq \Phi_0$$

with $$\left(\frac{r_0}{R_1} \right)^2 \ll 1$$

where r_0 is given by

$$\alpha = \Phi_0 \quad .$$

Then the shell is approximately representable by the osculating sphere at the apex. In fact if the surface of revolution $z = z(r)$ is expanded in a Maclaurin's series at the apex we have

$$z = \frac{1}{2A_0} r^2 + \frac{1}{6} (z''')_0 r^3 + \ldots$$

and furthermore if the shapes are limited to those for which the cross-sectional curves are of second degree (ellipsoids, parabolas) the approximations are provably satisfactory so that one may use the results of the theory of vibration of shallow spherical shells [2] to determine the approximate modal shapes and corresponding frequencies of vibration for shallow paraboloidal and ellipsoidal shells of revolution.

INEXTENSIONAL AND EXTENSIONAL VIBRATIONS

The theory of vibration which has been discussed so far may be greatly simplified by considering only the purely inextensional motions or the purely extensional motions. The inextensional type which depends on the assumption that the length of line elements remain invariant under deformation of the shell has been treated by Lin and Lee for the paraboloid [5]. The limitations of such assumptions are obvious

and that type of vibration will not be considered further
in the present report.

The theory of extensional vibrations is a more applicable theory, especially in the case of closed shells such as spheres [6]. Briefly, it may be noted that if the flexural couples G_1 , G_2 are taken identically zero or the flexural rigidity D is taken zero, Equations (10) reduce in complexity.

Since elastic free vibrations are harmonic in time, we may write the displacement functions

$$v(\alpha,t) = \sum v_n(\alpha) \cos (p_n t + \Theta)$$

$$w(\alpha,t) = \sum w_n(\alpha) \cos (p_n t + \Theta) \quad .$$

Then with the assumptions which have been specified the equations of motion may be written as

$$\alpha_1 \frac{d^2 v_n}{d\alpha^2} + \alpha_2 \frac{dv_n}{d\alpha} + \alpha_3 \frac{dw_n}{d\alpha} + \alpha_4 w_n$$

$$+ v_n(\alpha_5 + p_n^2 \sin \alpha \frac{\rho a^2 (1 - \nu^2)}{4E}) = 0$$

and $\quad \beta_1 \dfrac{dv_n}{d\alpha} + \beta_2 v_n + \beta_3 w_n = 0$

where α_i and β_i are relatively simple functions of α and β_3 also involves p_n .

Obviously w_n may be eliminated and one equation can be written for v_n as follows

$$A \frac{d^2 v}{d\alpha^2} + B \frac{dv}{d\alpha} + C v = 0$$

where A, B, C are rather complicated functions of α. However, since the coefficients are non-zero at the apex $(\alpha = 0)$ the point $\alpha = 0$ is a regular singular point and the solution is expandable in series about such a point.

As a practical matter however, it is more straight-forward to consider the two simultaneous equations for which series solutions may now be written

$$v = \sum A_n \phi^{n+S}$$

$$w = \sum B_n \phi^{n+S}$$

$$n = 0, 1, 2, \ldots$$

and S is determined to be ± 1 by usual methods [7]. It has been shown that a power series solution exists for $S = +1$ and not for $S = -1$.

Although straightforward the numerical calculations for specific cases are still rather excessive for illustrative purposes, especially when it is considered that the present needs are in connection with problems which necessarily involve flexural stresses.

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EXPERIMENTAL INVESTIGATION OF FLEXURAL VIBRATIONS
OF PARABOLOIDAL SHELLS OF REVOLUTION

Although solutions of the equations of motion for flexural vibrations were not obtained in the present study, it was considered to be instructive to perform experiments on two models of paraboloidal shells of revolution. The results may serve ultimately to indicate the type of functions necessary to serve as solutions to the equations of motion.

A sketch of the experimental model is shown in Fig. 2. The various parameters necessary for a complete description of the shell are provided. Also, the experimental apparatus are shown in Fig. 3. The shells were driven at resonant frequencies and the modal shapes painstakingly traced out with the aid of crystal-type transducers.

The modal shapes and corresponding frequencies are given in Tables No. 1, No. 2 and No. 3 for both fixed and free boundary conditions.

ACKNOWLEDGMENTS

-11-

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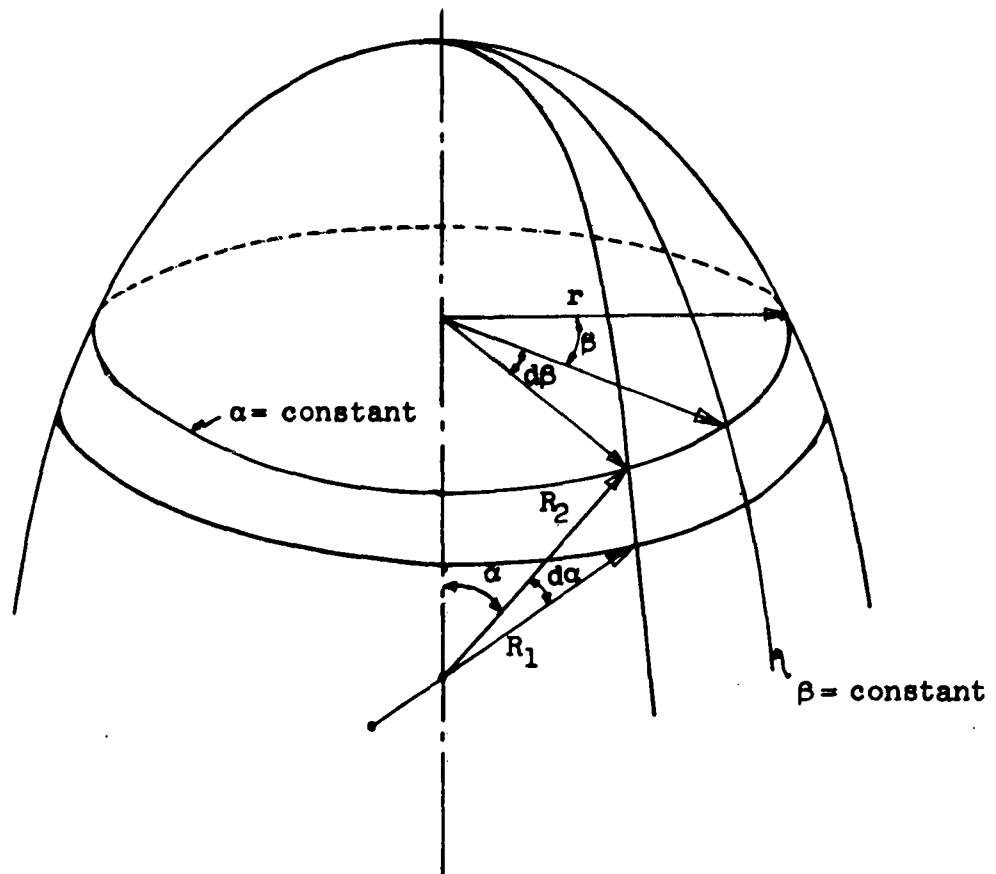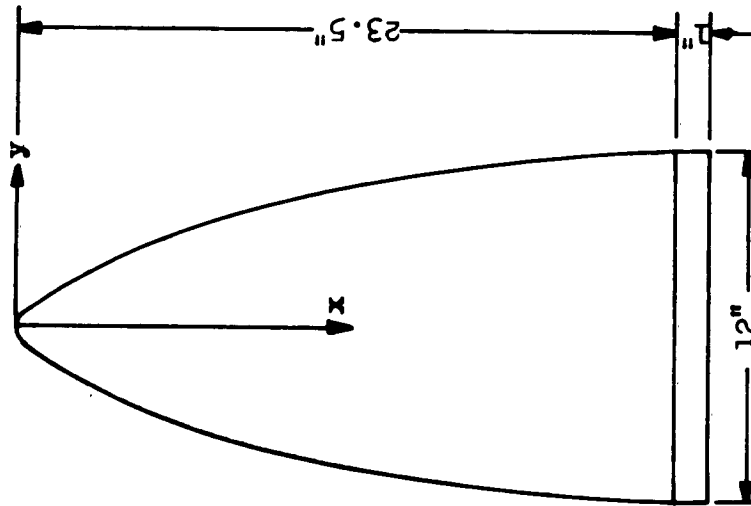7. E. L. Ince, Ordinary Differential Equations, Dover, New York, p. 396, (1956).
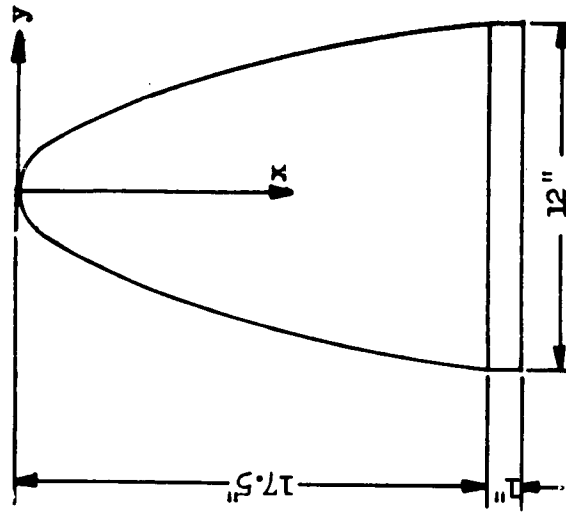
FIG. 1. PARABOLOIDAL SHELL OF REVOLUTION

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$y^2 = 2.024x$
Thickness = 0.063"
Weight = 8.75 lbs.

$y^2 = 1.408x$
Thickness = 0.060"
Weight = 12.25 lbs.

FIG. 2. EXPERIMENTAL MODELS OF SHELLS
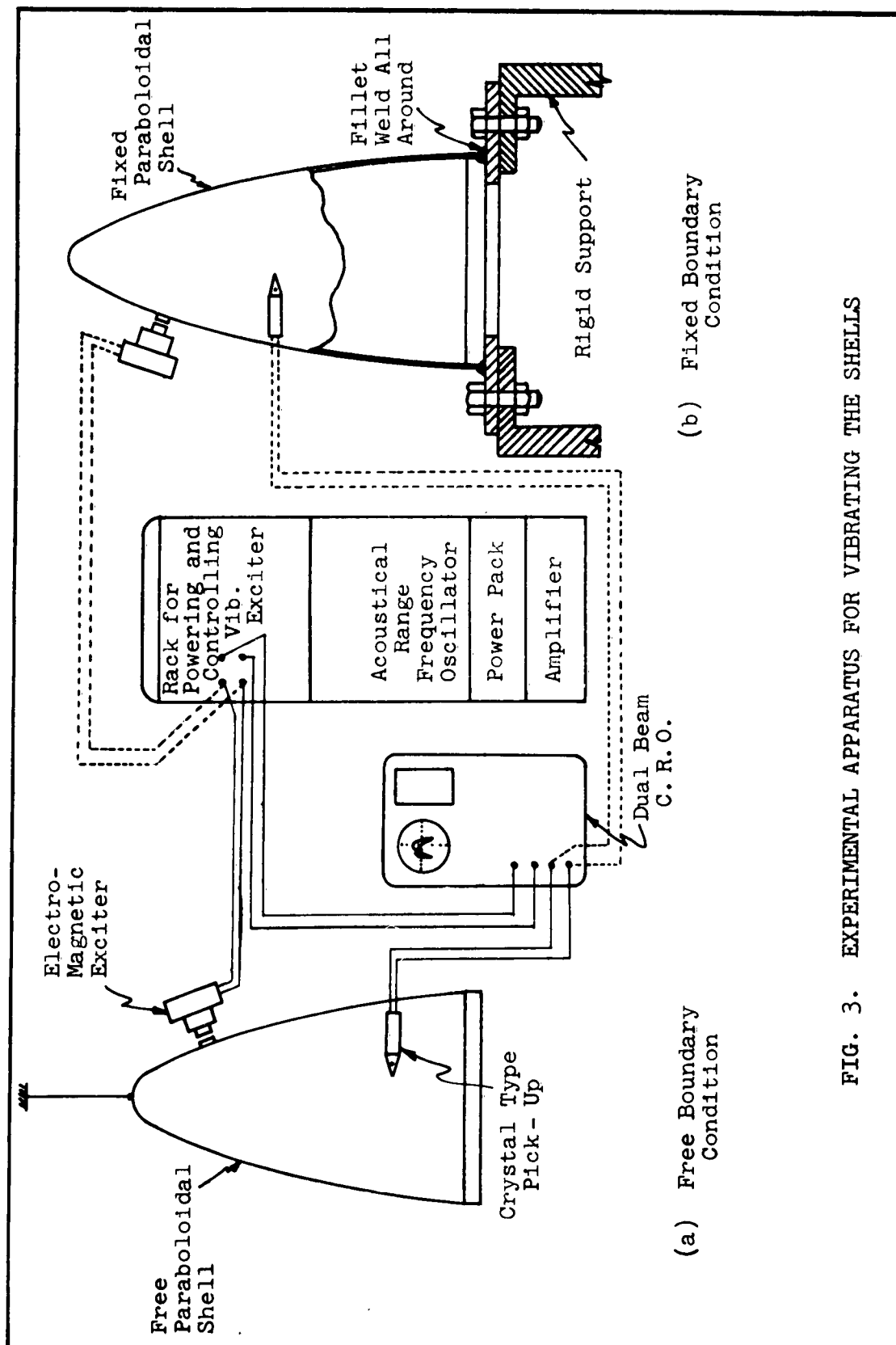
FIG. 3. EXPERIMENTAL APPARATUS FOR VIBRATING THE SHELLS

(a) Free Boundary Condition

(b) Fixed Boundary Condition

TABLE 1. NODAL PATTERNS AND FREQUENCIES FOR PARABOLOIDAL SHELL OF REVOLUTION

FREE BOUNDARY

SHELL HEIGHT = 24.5"

No. 1
70 cps

No. 2
180 cps

No. 3
330 cps

No. 4
524 cps

No. 5
744 cps

No. 6
4180 cps

No. 7
604 cps

No. 8
672 cps

No. 9
860 cps

No. 10
1080 cps

No. 11
2400 cps

No. 12
1180 cps

No. 13
1250 cps

No. 14
1480 cps

No. 15
4800 cps

No. 16
2920 cps

No. 17
2200 cps

No. 18
2610 cps

No. 19
4250 cps
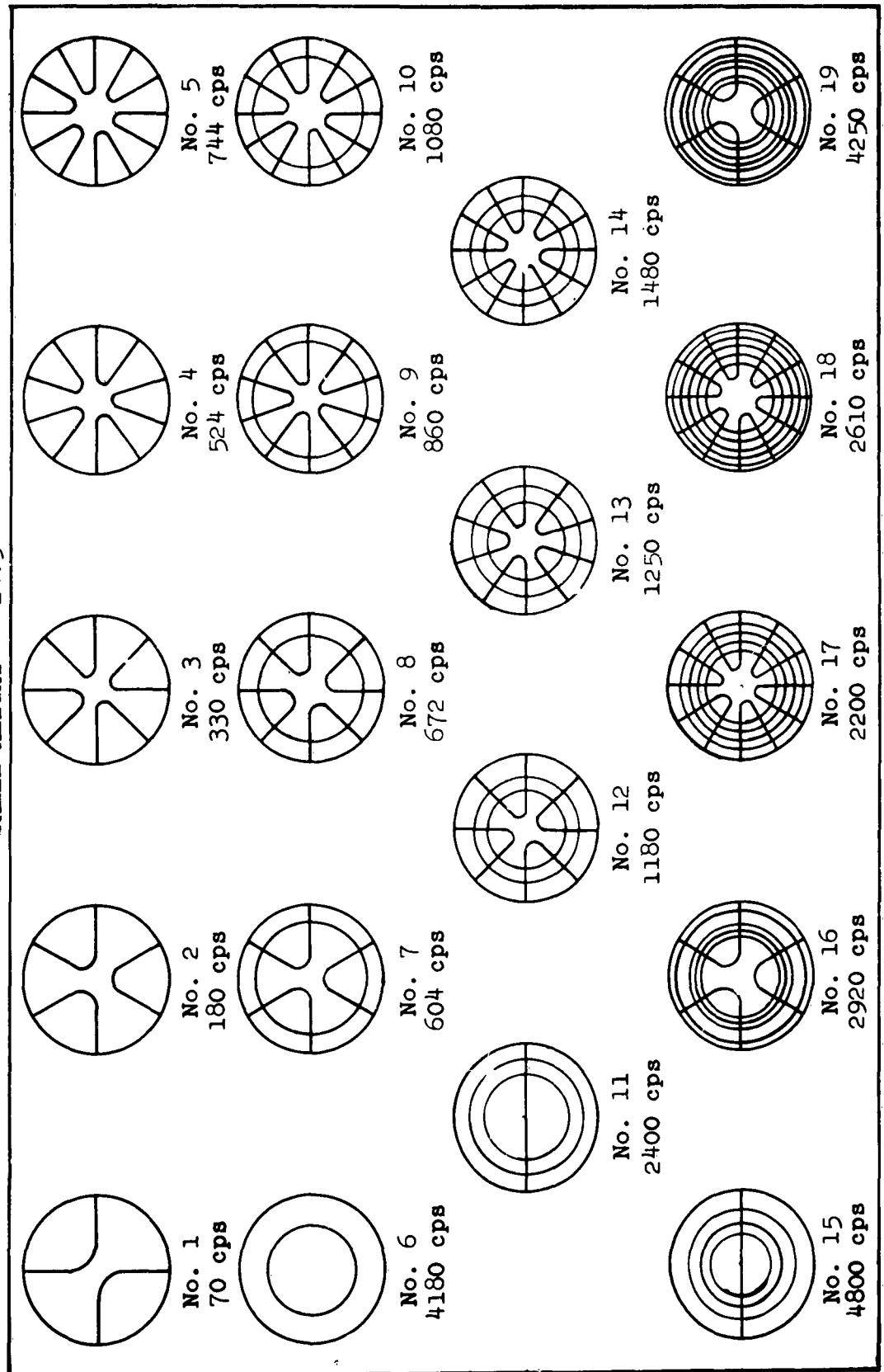
TABLE 2.   NODAL PATTERNS AND FREQUENCIES
FOR PARABOLOIDAL SHELL OF REVOLUTION

FIXED BOUNDARY
SHELL HEIGHT = 18.5"

No. 1	No. 2	No. 3	No. 4	No. 5	No. 6
1010 cps	820 cps	800 cps	890 cps	1070 cps	1740 cps

No. 7	No. 8
870 cps	780 cps

No. 9	No. 10	No. 11	No. 12	No. 13
2800 cps	2500 cps	1650 cps	1500 cps	1432 cps

No. 14	No. 15	No. 16	No. 17
2750 cps	2240 cps	2160 cps	2040 cps

No. 18	No. 19	No. 20
3040 cps	2600 cps	3200 cps

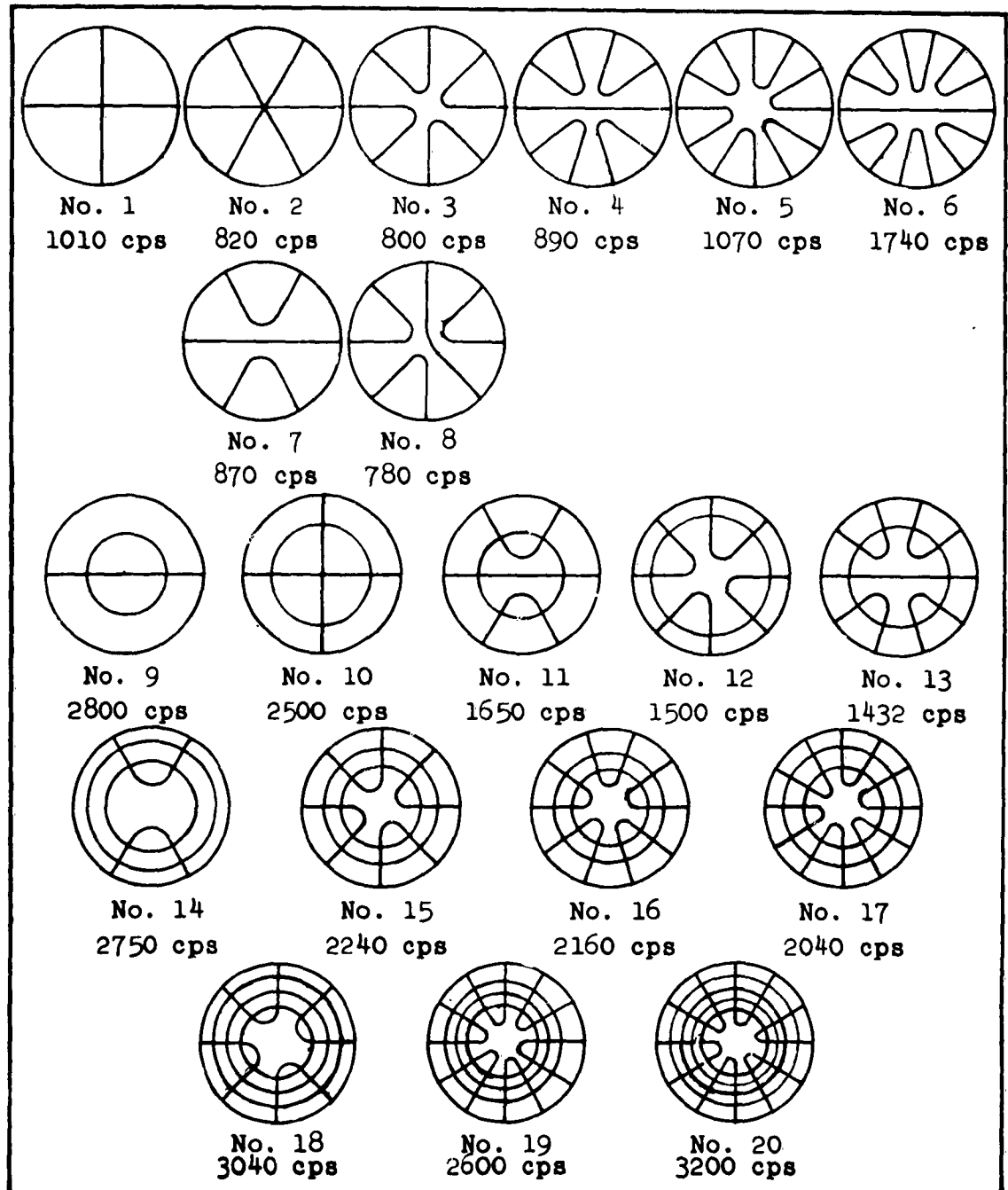TABLE 3. NODAL PATTERNS AND FREQUENCIES FOR PARABOLOIDAL SHELL OF REVOLUTION

FREE BOUNDARY

SHELL HEIGHT = 18.5"

No. 1
65 cps

No. 2
168 cps

No. 3
325 cps

No. 4
540 cps

No. 5
765 cps

No. 6
1200 cps